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The global solution for a class of systems of fractional nonlinear Schrödinger equations with periodic boundary condition

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ABSTRACT

In this paper, we consider a class of systems of fractional nonlinear Schrödinger equations. We prove the existence and uniqueness of the global solution to the periodic boundary value problem by using the Faedo–Galërkin method.

1. Introduction

Nonlinear Schrödinger equations have been used to analyze several physical situations, and have attracted the attention of researchers, especially in optics and hydrodynamics. In optics, systems of coupled nonlinear equations can be used to describe the propagation of light along birefringent optical fibers. A pair of light waves may move along the fiber, and the existence of each wave in the pair may depend on the cross phase modulation from the other. The phenomenon arises purely due to coupling, and is absent for a single mode [1]. In order to solve the nonlinear phenomenon about the optical solitons that store and transfer information in polyfibrous optical media, integrable or nearly integrable systems of two coupled nonlinear Schrödinger equations were considered [2]. For modeling real physical situations on the effect of linear and nonlinear damping (growth), a large number of experiments have demonstrated, in strongly dispersive media, that the nonlinear Schrödinger equation has an exact soliton solution that is valid under equilibrium conditions. We found that the system of coupled nonlinear Schrödinger equations can describe wave–wave interaction under the existence of another wave [3]. In hydrodynamics, nonlinear Schrödinger equations can be used to demonstrate the modulation instability of gravity waves in fluids at finite or great depths. However, the case of nonlinear Schrödinger equations is vague and intriguing [4]. In [5], intensive efforts have been made to search for special coupled nonlinear Schrödinger equations that pass standard tests for integrability, for dealing with higher order nonlinear effects.

However, more and more researchers found that integral order partial differential equations cannot be used to describe some real physical phenomenon exactly and that the appearance of fractional partial differential equations plays an important role in physics. Laskin [6,7] showed that the path integral over Lévy-like quantum mechanical paths allows a generalization of quantum mechanics. Namely, if the path integral over Brownian trajectories leads to the well-known Schrödinger equation, then the path integral over Lévy trajectories leads to the fractional Schrödinger equation. The fractional Schrödinger equation includes the space derivative of order α instead of the second ($\alpha = 1$) order space derivative in the standard Schrödinger equation. Afterward, the authors [8,9] studied some physical applications of the fractional Schrödinger equation and the generalized fractional Schrödinger equation with space time fractional derivatives. However, few theoretical analyses have been carried out for fractional nonlinear Schrödinger equations.

In this paper, we consider the following class of systems of fractional nonlinear Schrödinger equations:

$$iu_{1t} + (-\triangle)^{\alpha}u_1 + \beta(|u_1|^{\rho} + |u_2|^{\rho})u_1 + k_1(x,t)u_1 = f_1(x,t), \tag{1.1}$$

$$iu_{2t} + (-\triangle)^{\alpha}u_2 + \beta(|u_1|^{\rho} + |u_2|^{\rho})u_2 + k_2(x,t)u_2 = f_2(x,t) \tag{1.2}$$

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with the initial condition:

$$u_1(x,0) = u_0^1(x), \qquad u_2(x,0) = u_0^2(x) \tag{1.3}$$

and the periodic boundary condition:

$$u_j(x + 2\pi e_i, t) = u_j(x,t), \quad (j = 1, 2), \tag{1.4}$$

where $x \in \mathbb{R}^n$ and $t > 0$, $k_j(x,t)$ is a real-valued function, $f_j(x,t)$ is a complex-valued function, and

$$e_i = (0, \dots, 0, 1, 0, \dots, 0) \quad (i = 1, \dots, n)$$

is an orthonormal basis of $\mathbb{R}^n$. In (1.1) and (1.2), i is the imaginary unit, α is a positive fraction, $\beta \in \mathbb{R}$, $\beta \neq 0$, $\rho > 0$.

In [10–14], Boling Guo studied the initial value problem and the periodic boundary value problem for a class of systems of standard (nonfractional) nonlinear Schrödinger equations. In this paper, we prove the existence and uniqueness of the global solution to the periodic boundary value problem for a class of systems of fractional nonlinear Schrödinger equations by using the Faedo–Galërkin method. Theorems 2.1 and 4.1 are the new fundamental results for a class of systems of fractional nonlinear Schrödinger equations. We obtain the global existence of solution and weak solution to the periodic boundary value problem in Theorems 2.1 and 4.1, respectively.

2. Preliminaries and main result

Since u_j is a periodic function, we can express u_j by a Fourier series

$$u_j = \sum_{k\in\mathbb{Z}^n} a_{j,k} e^{i\langle k,x\rangle}, \quad (j = 1, 2). \tag{2.1}$$

Then

$$\partial_{x_h} u_j = \sum_{k\in\mathbb{Z}^n} i k_h a_{j,k} e^{i\langle k,x\rangle},$$

so $(-\triangle)^\alpha u_j$ is defined by

$$(-\triangle)^\alpha u_j = \sum_{k\in\mathbb{Z}^n} |k|^{2\alpha} a_{j,k} e^{i\langle k,x\rangle}. \tag{2.2}$$

Let

$$A = \left\{ \varphi \mid \varphi = \sum_{k\in\mathbb{Z}^n} b_k e^{i\langle k,x\rangle}, \sum_{k\in\mathbb{Z}^n} |k|^{4\alpha} b_k^2 < \infty, \sum_{k\in\mathbb{Z}^n} b_k^2 < \infty \right\}. \tag{2.3}$$

Let $H_{per}^{2\alpha}$ be a complete space of the set A under the norm

$$\|\varphi\|_{H_{per}^{2\alpha}} = \left(\sum_{k\in\mathbb{Z}^n} |k|^{4\alpha} b_k^2 \right)^{\frac{1}{2}} + \left(\sum_{k\in\mathbb{Z}^n} b_k^2 \right)^{\frac{1}{2}}.$$

Then $H_{per}^{2\alpha}$ is a Banach space. If φ and ψ belong to $H_{per}^{2\alpha}$, the combining (2.1)–(2.3) and Parseval's identity we conclude the following equation

$$\int_\Omega (-\triangle)^\alpha \varphi \cdot \psi \, dx = \int_\Omega (-\triangle)^{\alpha_1} \varphi \cdot (-\triangle)^{\alpha_2} \psi \, dx, \tag{2.4}$$

where α_1 and α_2 are nonnegative and $\alpha_1 + \alpha_2 = \alpha$.

Let $\Omega = (0, 2\pi) \times (0, 2\pi) \times \cdots \times (0, 2\pi) \subset \mathbb{R}^n$. Throughout this paper, we denote by $\|\cdot\|$ the norm of $H = L^2(\Omega)$ with usual inner product $(\cdot,\cdot)$, denote by $\|\cdot\|_{L^p(\Omega)}$ the norm of $L^p(\Omega)$ for all $1 \le p \le \infty$ ($\|\cdot\|_{L^2(\Omega)} = \|\cdot\|$). We denote by $H_{per}^{-\alpha}(\Omega)$ the dual space to $H_{per}^\alpha(\Omega)$. In order to study the problem (1.1)–(1.4), we introduce the Banach space $V = H_{per}^\alpha(\Omega) \cap L^{\rho+2}(\Omega)$ with norm

$$\|v\|_{H_{per}^\alpha(\Omega)\cap L^{\rho+2}(\Omega)} = \|v\|_{H_{per}^\alpha(\Omega)} + \|v\|_{L^{\rho+2}(\Omega)}.$$

Let X denote a Banach space, with norm $\|\cdot\|_X$. We need the following definitions.

Definition 2.1. The space $L^p(0, T; X)$ consists of all measurable functions $f : [0, T] \to X$ with

$$\|f\|_{L^p(0,T;X)} = \left(\int_0^T \|f\|_X^p \mathrm{d}t\right)^{\frac{1}{p}} < \infty,$$

for $1 \le p < \infty$, and

$$\|f\|_{L^\infty(0,T;X)} = \lim_{0\le t\le T} \sup \|f(t)\|_X < \infty.$$

Definition 2.2. The space $C([0, T]; X)$ comprises all continuous functions $f : [0, T] \to X$ with

$$\|f\|_{C([0,T];X)} = \max_{0\le t\le T} \|f(t)\| < \infty.$$

We have the following main result.

Theorem 2.1. *Let $\alpha > \frac{n}{2}$. If ρ is an even, suppose that $0 < \rho < \frac{4\alpha}{n}$. If ρ is not an even, suppose that $2[\alpha] + 1 < \rho < \frac{4\alpha}{n}$. Assume that $k_j(x, t) \in C^2 \cap H^{2\alpha}_{per}(\Omega)$, k_{jt} and k_{jtt} are bounded, $f_j(x, t) \in L^2 \cap H^{2\alpha}_{per}(\Omega)$ and $f_{jt}, f_{jtt} \in L^2(\Omega) (j = 1, 2)$. Then for all $u_0^1 \in H^{4\alpha}_{per}(\Omega)$, $u_0^2 \in H^{4\alpha}_{per}(\Omega)$, there exists a unique global solution $u = (u_1, u_2)$ of the problem (1.1)–(1.4), such that*

$$u_j \in L^\infty(0, T; H^{4\alpha}_{per}(\Omega)), \qquad u_{jt} \in L^\infty(0, T; H^{2\alpha}_{per}(\Omega)), \quad (j = 1, 2).$$

Theorem 2.1 generalizes the result for a class of systems of nonlinear Schrödinger equations in [10].

3. A priori estimates

We can construct the global approximate solution by the Galërkin method. In order to prove the convergence of these approximate solutions, we need to obtain a priori estimates of these approximate solutions. Therefore we first prove a priori estimates of the problem (1.1)–(1.4), these estimates are same as a priori estimates of Galërkin approximate solutions.

Lemma 3.1. *Suppose that $\alpha > 0$ and $\rho > 0$, $u = (u_1, u_2)$ solves the problem (1.1)–(1.4), and $\|f_j\|^2 \leqslant M_1$ $(j = 1, 2)$. Then*

$$\sup_{0\le t<T} \sum_{j=1}^{2} \|u_j\| \leqslant E_0, \tag{3.1}$$

where the constant E_0 depends only on M_1, $\|u_j(x, 0)\|$ and T.

Proof. Taking the inner product of (1.1) with $\bar{u}_1$ and the inner product of (1.2) with $\bar{u}_2$ respectively, then taking the imaginary part to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_j\|^2 = \mathrm{Im} \int_\Omega 2f_j\bar{u}_j \mathrm{d}x, \quad j = 1, 2.$$

Applying the Hölder inequality to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_j\|^2 \leqslant 2\int_\Omega |f_j u_j| \mathrm{d}x \leqslant \|f_j\|^2 + \|u_j\|^2 \le M_1 + \|u_j\|^2, \quad j = 1, 2. \tag{3.2}$$

Then by using Gronwall inequality, we obtain (3.1). □

Here, and throughout, we use T to denote an arbitrary positive constant.

Lemma 3.2. *Let $\alpha > \frac{n}{2}$. Suppose that $0 < \rho < \frac{4\alpha}{n}$, $u = (u_1, u_2)$ solves the problem (1.1)–(1.4), and assume $\|f_j\|^2 \leqslant M_1$, $|k_j| \leqslant M_2$, $|k_{jt}| \leqslant M_3$, $\|f_{jt}\|^2 \leqslant M_4 (j = 1, 2)$. Then,*

$$\sup_{0\le t<T} \sum_{j=1}^{2} \|(-\triangle)^{\frac{\alpha}{2}} u_j\| \le E_1, \tag{3.3}$$

where the constant E_1 depends on $\|u_j(x, 0)\|_{H^\alpha_{per}(\Omega)}$, $\|u_j(x, 0)\|_{L^{\rho+2}(\Omega)}$, M_1, M_2, M_3, M_4 and T.

Proof. Taking the inner product of (1.1) with $\bar{u}_{1t}$ and taking the inner product of (1.2) with $\bar{u}_{2t}$ respectively, we obtain

$$(\mathrm{i}u_{jt}, u_{jt}) + ((-\triangle)^\alpha u_j, u_{jt}) + (\beta(|u_1|^\rho + |u_2|^\rho)u_j, u_{jt}) + (k_j u_j, u_{jt}) = (f_j, u_{jt}).$$

Taking the real part of the above equality to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |(-\triangle)^{\frac{\alpha}{2}}u_j|^2\mathrm{d}x + \int_\Omega \left(\beta(|u_1|^\rho + |u_2|^\rho)\frac{\mathrm{d}}{\mathrm{d}t}|u_j|^2\right)\mathrm{d}x + \int_\Omega \left(k_j\frac{\mathrm{d}}{\mathrm{d}t}|u_j|^2\right)\mathrm{d}x = \int_\Omega (f_j\bar{u}_{jt} + \bar{f}_j u_{jt})\mathrm{d}x.$$

So

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\sum_{j=1}^{2}|(-\triangle)^{\frac{\alpha}{2}}u_j|^2 + \beta(|u_1|^\rho + |u_2|^\rho)(|u_1|^2+|u_2|^2)\right)\mathrm{d}x\\
&= \sum_{j=1}^{2}\left(\int_\Omega (f_j\bar{u}_{jt} + \bar{f}_j u_{jt})\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (k_j|u_j|^2)\mathrm{d}x + \int_\Omega (k_{jt}|u_j|^2)\mathrm{d}x\right) + \int_\Omega \beta(|u_1|^2+|u_2|^2)\frac{\mathrm{d}}{\mathrm{d}t}(|u_1|^\rho + |u_2|^\rho)\mathrm{d}x\\
&\le C + \frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{2}\left(\int_\Omega ((f_j\bar{u}_j + \bar{f}_j u_j) - k_j|u_j|^2)\mathrm{d}x\right)\\
&\quad + \int_\Omega \frac{\beta\rho}{2}\left(|u_1|^\rho\frac{\mathrm{d}}{\mathrm{d}t}|u_1|^2 + |u_2|^\rho\frac{\mathrm{d}}{\mathrm{d}t}|u_2|^2 + |u_1|^{\rho-2}|u_2|^2\frac{\mathrm{d}}{\mathrm{d}t}|u_1|^2 + |u_2|^{\rho-2}|u_1|^2\frac{\mathrm{d}}{\mathrm{d}t}|u_2|^2\right)\mathrm{d}x\\
&\le c + \frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{2}\left(\int_\Omega ((f_j\bar{u}_j + \bar{f}_j u_j) - k_j|u_j|^2)\mathrm{d}x\right) + c\|u_1\|^\rho_{L^\infty(\Omega)}\\
&\quad + c\|u_2\|^\rho_{L^\infty(\Omega)} + c\|u_1\|^{\rho-2}_{L^\infty(\Omega)}\|u_2\|^2_{L^\infty(\Omega)} + c\|u_2\|^{\rho-2}_{L^\infty(\Omega)}\|u_1\|^2_{L^\infty(\Omega)}.
\end{aligned}\tag{3.4}$$

Integrating the above inequality with respect to t, we have

$$\begin{aligned}
&\int_\Omega \sum_{j=1}^{2}|(-\triangle)^{\frac{\alpha}{2}}u_j|^2 + \beta(|u_1|^\rho + |u_2|^\rho)(|u_1|^2+|u_2|^2)\mathrm{d}x\\
&\le c + \sum_{j=1}^{2}\|(-\triangle)^{\frac{\alpha}{2}}u_j(x,0)\|^2 + \int_\Omega \beta(|u_1(x,0)|^\rho + |u_2(x,0)|^\rho)(|u_1(x,0)|^2 + |u_2(x,0)|^2)\mathrm{d}x\\
&\quad + c\int_0^t (\|u_2\|^{\rho-2}_{L^\infty(\Omega)}\|u_1\|^2_{L^\infty(\Omega)} + \|u_1\|^{\rho-2}_{L^\infty(\Omega)}\|u_2\|^2_{L^\infty(\Omega)})\mathrm{d}t + c\int_0^t (\|u_1\|^\rho_{L^\infty(\Omega)} + \|u_2\|^\rho_{L^\infty(\Omega)})\mathrm{d}t.
\end{aligned}\tag{3.5}$$

Let $a = \frac{n}{2\alpha} < 1$, Then

$$0 = a\left(\frac{1}{2} - \frac{\alpha}{n}\right) + \frac{1-a}{2}.$$

So by Gagliardo–Nirenberg inequality, we have

$$\|u_j\|^\rho_{L^\infty(\Omega)} \le c\|(-\triangle)^{\frac{\alpha}{2}}u_j\|^{\frac{n\rho}{2\alpha}}\tag{3.6}$$

$$\|u_i\|^{\rho-2}_{L^\infty(\Omega)}\|u_j\|^2_{L^\infty(\Omega)} \le c\|(-\triangle)^{\frac{\alpha}{2}}u_i\|^{\frac{n(\rho-2)}{2\alpha}}\|(-\triangle)^{\frac{\alpha}{2}}u_j\|^{\frac{n}{\alpha}},\tag{3.7}$$

since $\rho < \frac{4\alpha}{n}$, $\frac{n\rho}{2\alpha} < 2$. Then from the above inequality, we get

$$\|u_j\|^\rho_{L^\infty(\Omega)} \le \varepsilon\|(-\triangle)^{\frac{\alpha}{2}}u_j\|^2 + c,$$

$$c\|u_i\|^{\rho-2}_{L^\infty(\Omega)}\|u_j\|^2_{L^\infty(\Omega)} \le c + \frac{\|(-\triangle)^{\frac{\alpha}{2}}u_i\|^2}{\frac{4\alpha}{n(\rho-2)}} + \frac{\|(-\triangle)^{\frac{\alpha}{2}}u_j\|^2}{\frac{2\alpha}{n}}.$$

Then (3.5) becomes

$$\sum_{j=1}^{2}\|(-\triangle)^{\frac{\alpha}{2}}u_j\|^2 + \int_\Omega \beta(|u_1|^\rho + |u_2|^\rho)(|u_1|^2+|u_2|^2)\mathrm{d}x \le c + c\int_0^t\sum_{j=1}^{2}\|(-\triangle)^{\frac{\alpha}{2}}u_j\|^2\mathrm{d}t,\tag{3.8}$$

where the constant c depends on $\|u_j(x,0)\|_{H^\alpha_{per}(\Omega)}$,$\|u_j(x,0)\|_{L^{\rho+2}(\Omega)}$, M_1, M_2, M_3, M_4, E_0 and T. Let $\delta = \frac{\rho n}{2(\rho+2)\alpha} < 1$. Then

$$\frac{1}{\rho+2} = \delta\left(\frac{1}{2} - \frac{\alpha}{n}\right) + \frac{1-\delta}{2}.$$

So by Gagliardo–Nirenberg inequality, we have

$$\|u_j\|^{\rho+2}_{L^{\rho+2}(\Omega)} \leq c\|(-\triangle)^{\frac{\alpha}{2}}u_j\|^{\frac{\rho n}{2\alpha}} \leq \varepsilon\|(-\triangle)^{\frac{\alpha}{2}}u_j\|^2 + c. \tag{3.9}$$

By using the above inequalities, we get

$$\sum_{j=1}^{2}\|(-\triangle)^{\frac{\alpha}{2}}u_j\|^2 \leq c\int_0^t\sum_{j=1}^{2}\|(-\triangle)^{\frac{\alpha}{2}}u_j\|^2\mathrm{d}t + c. \tag{3.10}$$

By Gronwall inequality, we obtain (3.3). □

Lemma 3.3. *Let $\alpha > \frac{n}{2}$. Suppose that ρ satisfies the conditions of Lemma 3.2, $u = (u_1, u_2)$ solves the problem (1.1)–(1.4). Then*

$$\sup_{0\leq t<T}\sum_{j=1}^{2}(\|u_{jt}\| + \|(-\triangle)^{\alpha}u_j\|) \leq E_2, \tag{3.11}$$

where the constant E_2 depends only on $\|u_j(x,0)\|_{H^{2\alpha}_{per}(\Omega)}$, M_1, M_2, M_3, M_4 and T.

Proof. Differentiating (1.1) with respect to t, taking the inner product with $\bar{u}_{1t}$, and differentiating (1.2) with respect to t, taking the inner product with $\bar{u}_{2t}$, respectively, we obtain

$$(\mathrm{i}u_{jtt}, u_{jt}) + ((-\triangle)^{\alpha}u_{jt}, u_{jt}) + \left(\frac{\mathrm{d}}{\mathrm{d}t}(\beta(|u_1|^{\rho} + |u_2|^{\rho})u_j), u_{jt}\right) + (k_ju_{jt}, u_{jt}) + (k_{jt}u_j, u_{jt}) = (f_{jt}, u_{jt}).$$

By taking the imaginary part of the above equality, we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{jt}\|^2 &+ \mathrm{Im}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\beta(|u_1|^{\rho} + |u_2|^{\rho})u_j), u_{jt}\right) + \frac{1}{2i}\int_{\Omega}(k_{jt}u_j\bar{u}_{jt} - k_{jt}\bar{u}_ju_{jt})\mathrm{d}x \\ &= \frac{1}{2i}\int_{\Omega}(f_{jt}\bar{u}_{jt} - \bar{f}_{jt}u_{jt})\mathrm{d}x.\end{aligned} \tag{3.12}$$

But

$$\begin{aligned}\mathrm{Im}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\beta(|u_1|^{\rho} + |u_2|^{\rho})u_j), u_{jt}\right) &= \mathrm{Im}\int_{\Omega}\frac{\rho\beta}{2}|u_1|^{\rho-2}u_j\bar{u}_{jt}(u_1\bar{u}_{1t} + u_{1t}\bar{u}_1)\mathrm{d}x \\ &\quad + \mathrm{Im}\int_{\Omega}\frac{\rho\beta}{2}|u_2|^{\rho-2}u_j\bar{u}_{jt}(u_2\bar{u}_{2t} + u_{2t}\bar{u}_2)\mathrm{d}x.\end{aligned} \tag{3.13}$$

Then by using (3.12) and (3.13), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{jt}\|^2 + \mathrm{Im}\int_{\Omega}\frac{\rho\beta}{2}|u_1|^{\rho-2}u_j\bar{u}_{jt}(u_1\bar{u}_{1t} + \bar{u}_1u_{1t})\mathrm{d}x + \mathrm{Im}\int_{\Omega}\frac{\rho\beta}{2}|u_2|^{\rho-2}u_j\bar{u}_{jt}(u_2\bar{u}_{2t} + \bar{u}_2u_{2t})\mathrm{d}x \leqslant C\|u_{jt}\|^2. \tag{3.14}$$

Since $\alpha > \frac{n}{2}$, $\|u_j\|_{L^{\infty}(\Omega)} \leq C\|u_j\|_{H^{\alpha}_{per}(\Omega)} \leq C$. Also we have

$$\begin{aligned}\mathrm{Im}\int_{\Omega}\left(\frac{\rho\beta}{2}|u_i|^{\rho-2}u_j\bar{u}_{jt}u_i\bar{u}_{it}\right)\mathrm{d}x &\leq C\|u_i\|^{\rho-1}_{L^{\infty}(\Omega)}\int_{\Omega}(|u_ju_{it}u_{jt}|)\mathrm{d}x \leq C\|u_j\|_{L^{\infty}(\Omega)}\int_{\Omega}|u_{it}u_{jt}|\mathrm{d}x \\ &\leq C(\|u_{it}\|^2 + \|u_{jt}\|^2), \quad (i = 1, 2).\end{aligned}$$

From (3.14) and the above inequality, we get

$$\frac{1}{2}\sum_{j=1}^{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{jt}\|^2 \leq \sum_{j=1}^{2}C\|u_{jt}\|^2.$$

Integrating the above inequality from 0 to t, we have

$$\sum_{j=1}^{2}\|u_{jt}\|^2 \leqslant \sum_{j=1}^{2}\|u_{jt}(x,0)\|^2 + C\sum_{j=1}^{2}\int_0^t\|u_{jt}\|^2\mathrm{d}s. \tag{3.15}$$

By applying (1.1) and (1.2), we have

$$\begin{aligned}\|u_{jt}(x,0)\| &\leq C\|(-\triangle)^{\alpha}u_j(x,0)\| + C\|\beta(|u_1(x,0)|^{\rho} + |u_2(x,0)|^{\rho})u_j(x,0)\| + C\|u_j(x,0)\| + C \\ &\leq C(\|u_j(x,0)\|_{H^{2\alpha}_{per}(\Omega)}).\end{aligned}$$

Then from (3.15), we get

$$\sum_{j=1}^{2}\|u_{jt}\|^2 \le C\sum_{j=1}^{2}\left(\int_0^t \|u_{jt}\|^2 \mathrm{d}s + \|u_j(x,0)\|_{H^{2\alpha}_{per}(\Omega)}\right).$$

In term of Gronwall inequality, we deduce that

$$\sum_{j=1}^{2}\|u_{jt}\|^2 \le C\sum_{j=1}^{2}(\|u_j(x,0)\|_{H^{2\alpha}_{per}(\Omega)}).$$

By (1.1) and the above inequality, we obtain (3.11). □

Lemma 3.4. *Let $\alpha > \frac{n}{2}$. Suppose that ρ satisfies the conditions of Lemma 3.2 if ρ is an even. If ρ is not an even, suppose that $2[\alpha]+1 < \rho < \frac{4\alpha}{n}$. Assume that $u=(u_1,u_2)$ solves the problem (1.1)–(1.4), and assume that $\|f_{jtt}\| \le M_5$, $|k_{jtt}| \le M_6$, $\|f_j(x,t)\|_{H^{2\alpha}_{per}(\Omega)} \le M_7$ and $\|k_j(x,t)\|_{H^{2\alpha}_{per}(\Omega)} \le M_8$. Then*

$$\sup_{0\le t<T}\sum_{j=1}^{2}(\|(-\triangle)^{\frac{\alpha}{2}}u_{jt}\| + \|u_{jtt}\| + \|(-\triangle)^{\alpha}u_{jt}\|) \le E_3, \tag{3.16}$$

where the constant E_3 depends on $\|u_j(x,0)\|_{H^{4\alpha}_{per}(\Omega)}$, M_1, M_2, M_3, M_4, M_7, M_8 and T.

Proof. Differentiating (1.1) with respect to t two times, taking the inner product with $\bar{u}_{1tt}$, and differentiating (1.2) with respect to t two times, taking the inner product with $\bar{u}_{2tt}$, respectively, we obtain

$$(\mathrm{i}u_{jttt}, u_{jtt}) + ((-\triangle)^{\alpha}u_{jtt}, u_{jtt}) + \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\beta(|u_1|^{\rho}+|u_2|^{\rho})u_j), u_{jtt}\right) + \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}(k_ju_j), u_{jtt}\right) = (f_{jtt}, u_{jtt}).$$

By taking the imaginary part of the above equality, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{jtt}\|^2 + \mathrm{Im}\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\beta(|u_1|^{\rho}+|u_2|^{\rho})u_j), u_{jtt}\right) + \mathrm{Im}\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}(k_ju_j), u_{jtt}\right) = \mathrm{Im}(f_{jtt}, u_{jtt}). \tag{3.17}$$

But

$$\begin{aligned}
&\mathrm{Im}\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\beta(|u_1|^{\rho}+|u_2|^{\rho})u_j), u_{jtt}\right)\\
&\quad= \mathrm{Im}(\rho\beta|u_1|^{\rho-2}(u_1\bar{u}_{1t}+\bar{u}_1u_{1t})u_{jt}, u_{jtt}) + \mathrm{Im}(\rho\beta|u_2|^{\rho-2}(u_2\bar{u}_{2t}+\bar{u}_2u_{2t})u_{jt}, u_{jtt})\\
&\qquad+\mathrm{Im}\frac{\rho\beta}{2}(|u_1|^{\rho-2}(u_1\bar{u}_{1tt}+u_{1tt}\bar{u}_1)u_j, u_{jtt}) + \mathrm{Im}\frac{\rho\beta}{2}(|u_2|^{\rho-2}(u_2\bar{u}_{2tt}+u_{2tt}\bar{u}_2)u_j, u_{jtt})\\
&\qquad+\mathrm{Im}\frac{\rho\beta(\rho-2)}{4}(u_j(|u_1|^{\rho-4}(u_1\bar{u}_{1t}+u_{1t}\bar{u}_1)^2+|u_2|^{\rho-4}(u_2\bar{u}_{2t}+u_{2t}\bar{u}_2)^2), u_{jtt})\\
&\qquad+\mathrm{Im}(\rho\beta|u_1|^{\rho-2}u_j|u_{1t}|^2, u_{jtt}) + \mathrm{Im}(\rho\beta|u_2|^{\rho-2}u_j|u_{2t}|^2, u_{jtt}).
\end{aligned} \tag{3.18}$$

For the first term and the second term to the right hand side of (3.18), we get

$$\begin{aligned}
\mathrm{Im}(\rho\beta|u_i|^{\rho-2}(u_i\bar{u}_{it}+\bar{u}_iu_{it})u_{jt}, u_{jtt}) &\le C\int_{\Omega}|u_i|^{\rho-1}|u_{it}u_{jt}u_{jtt}|\mathrm{d}x\\
&\le C\|u_{it}\|_{L^4(\Omega)}\|u_{jt}\|_{L^4(\Omega)}\|u_{jtt}\|\\
&\le C\|u_{jtt}\|^2 + \varepsilon\|u_{it}\|^4_{L^4(\Omega)} + \varepsilon\|u_{jt}\|^4_{L^4(\Omega)}, \quad (i=1,2),
\end{aligned} \tag{3.19}$$

where ε is a arbitrary small constant.

As the same reason, for the third term and the fourth term to the right hand side of (3.18), we have

$$\begin{aligned}
\mathrm{Im}\frac{\rho\beta}{2}(|u_i|^{\rho-2}(u_i\bar{u}_{itt}+u_{itt}\bar{u}_i)u_j, u_{jtt}) &\le C\int_{\Omega}|u_i|^{\rho-1}|u_{itt}u_ju_{jtt}|\mathrm{d}x\\
&\le C(\|u_{itt}\|^2+\|u_{jtt}\|^2), \quad (i=1,2).
\end{aligned} \tag{3.20}$$

For the fifth term and the sixth term to the right hand side of (3.18), we have

$$\mathrm{Im}\frac{\rho\beta(\rho-2)}{4}(u_j(|u_1|^{\rho-4}(u_1\bar{u}_{1t}+u_{1t}\bar{u}_1)^2+|u_2|^{\rho-4}(u_2\bar{u}_{2t}+u_{2t}\bar{u}_2)^2), u_{jtt})\mathrm{d}x$$

$$\leq C\int_{\Omega}(|u_j||u_1|^{\rho-4}|u_1|^2|u_{1t}|^2|u_{jtt}|+|u_j||u_2|^{\rho-4}|u_2|^2|u_{2t}|^2|u_{jtt}|)$$
$$\leq C\|u_{jtt}\|^2+\varepsilon\|u_{1t}\|^4_{L^4(\Omega)}+\varepsilon\|u_{2t}\|^4_{L^4(\Omega)}. \tag{3.21}$$

For the seventh term and the last term to the right hand side of (3.18), we get

$$\mathrm{Im}(\rho\beta|u_1|^{\rho-2}u_j|u_{1t}|^2+\rho\beta|u_2|^{\rho-2}u_j|u_{2t}|^2,u_{jtt})\leqslant C\|u_{jtt}\|^2+\varepsilon\|u_{1t}\|^4_{L^4(\Omega)}+\varepsilon\|u_{2t}\|^4_{L^4(\Omega)}. \tag{3.22}$$

For the third term to the left hand side of (3.17), we have

$$\mathrm{Im}\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}(k_ju_j),u_{jtt}\right)\leqslant C\int_{\Omega}(|u_ju_{jtt}|+|u_{jt}u_{jtt}|)\mathrm{d}x+C\|u_{jtt}\|^2\leqslant C\|u_{jtt}\|^2. \tag{3.23}$$

For the term to the right hand side of (3.17), we have

$$\mathrm{Im}(f_{jtt},u_{jtt})\leqslant\int_{\Omega}|f_{jtt}u_{jtt}|\mathrm{d}x\leqslant C+C\|u_{jtt}\|^2. \tag{3.24}$$

Then by (3.17)–(3.24), we conclude that

$$\sum_{j=1}^2\|u_{jtt}\|^2\leq\varepsilon\sum_{j=1}^2\int_0^t\|u_{jt}\|^4_{L^4}\mathrm{d}s+C\sum_{j=1}^2\int_0^t\|u_{jtt}\|^2\mathrm{d}s+\sum_{j=1}^2\|u_{jtt}(x,0)\|^2+C. \tag{3.25}$$

Let $\theta=\frac{n}{8\alpha}<\frac{1}{4}$. Then

$$\frac{1}{4}=\theta\left(\frac{1}{2}-\frac{\alpha}{n}\right)+(1-\theta)\frac{1}{2}.$$

By (3.11) and Gagliardo–Nirenberg inequality, we have

$$\|u_{jt}\|_{L^4(\Omega)}\leq C\|u_{jt}\|^{1-\theta}\|(-\triangle)^{\frac{\alpha}{2}}u_{jt}\|^{\theta}\leq C\|(-\triangle)^{\frac{\alpha}{2}}u_{jt}\|^{\theta}. \tag{3.26}$$

By (1.1), (1.2) and (3.11), we get

$$\begin{aligned}\|u_{jtt}(x,0)\|&\leq\|(-\triangle)^{\alpha}((-\triangle)^{\alpha}u_j(x,0))+(\beta(|u_1(x,0)|^{\rho}+|u_2(x,0)|^{\rho})u_j(x,0)+k_j(x,0)u_j(x,0)-f_j(x,0))\|\\&\quad+\left\|\frac{\mathrm{d}}{\mathrm{d}t}(\beta(|u_1(x,0)|^{\rho}+|u_2(x,0)|^{\rho})u_j(x,0))\right\|+\left\|\frac{\mathrm{d}}{\mathrm{d}t}(k_j(x,0)u_j(x.0))\right\|+\|f_{jt}(x,0)\|\\&\leq C\|u_j(x,0)\|_{H^{4\alpha}_{per}(\Omega)}+C\|(-\triangle)^{\alpha}(|u_1(x,0)|^{\rho}u_j(x,0)+|u_2(x,0)|^{\rho}u_j(x,0))\|+C.\end{aligned} \tag{3.27}$$

If $\alpha\geq\max\{\frac{n}{2},1\}$,

$$\begin{aligned}\|(-\triangle)^{\alpha}(|u_1(x,0)|^{\rho}u_j(x,0))\|&\leq C\|(-\triangle)^{[\alpha]+1}(|u_1(x,0)|^{\rho}u_j(x,0))\|\\&\leq C(\|u_1(x,0)\|_{H^{4\alpha}_{per}(\Omega)}+\|u_2(x,0)\|_{H^{4\alpha}_{per}(\Omega)}),\end{aligned} \tag{3.28}$$

where we use the condition $\rho>2[\alpha]+1$ if ρ is not an even.

If $n=1$ and $\frac{1}{2}<\alpha<1$,

$$\|(-\triangle)^{\alpha}(|u_1(x,0)|^{\rho}u_j(x,0))\|\leq C(\|u_1(x,0)\|_{H^{4\alpha}_{per}(\Omega)}+\|u_2(x,0)\|_{H^{4\alpha}_{per}(\Omega)}). \tag{3.29}$$

Hence, from (3.27)–(3.29), we derive that

$$\|u_{jtt}(x,0)\|\leq C(\|u_1(x,0)\|_{H^{4\alpha}_{per}(\Omega)}+\|u_2(x,0)\|_{H^{4\alpha}_{per}(\Omega)}). \tag{3.30}$$

From (3.25) and (3.30), we deduce that

$$\sum_{j=1}^2\|u_{jtt}\|^2\leq C\sum_{j=1}^2\int_0^t\|u_{jtt}\|^2\mathrm{d}s+C\sum_{j=1}^2\|u_j(x,0)\|_{H^{4\alpha}_{per}(\Omega)}+\varepsilon\sum_{j=1}^2\int_0^t\|(-\triangle)^{\frac{\alpha}{2}}u_{jt}\|^2\mathrm{d}s. \tag{3.31}$$

Differentiating (1.1) with respect to t, taking the inner product with $\bar{u}_{1t}$, and differentiating (1.2) with respect to t, taking the inner product with, respectively, we obtain

$$(\mathrm{i}u_{jtt},u_{jt})+((-\triangle)^{\alpha}u_{jt},u_{jt})+\left(\frac{\mathrm{d}}{\mathrm{d}t}(\beta(|u_1|^{\rho}+|u_2|^{\rho})u_j),u_{jt}\right)+\left(\frac{\mathrm{d}}{\mathrm{d}t}(k_ju_j),u_{jt}\right)=(f_{jt},u_{jt}). \tag{3.32}$$

By taking the real part of the above inequality, we get

$$\|(-\triangle)^{\frac{\alpha}{2}}u_{jt}\|^2 = -\mathrm{Re}(\mathrm{i}u_{jtt}, u_{jt}) - \mathrm{Re}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\beta(|u_1|^\rho + |u_2|^\rho)u_j), u_{jt}\right) - \mathrm{Re}\left(\frac{\mathrm{d}}{\mathrm{d}t}(k_ju_j), u_{jt}\right) + \mathrm{Re}(f_{jt}, u_{jt}). \tag{3.33}$$

But

$$-\mathrm{Re}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\beta(|u_1|^\rho + |u_2|^\rho)u_j), u_{jt}\right) - \mathrm{Re}\left(\frac{\mathrm{d}}{\mathrm{d}t}(k_ju_j), u_{jt}\right) + \mathrm{Re}(f_{jt}, u_{jt}) \le C. \tag{3.34}$$

From (3.33), we get

$$\|(-\triangle)^{\frac{\alpha}{2}}u_{jt}\|^2 \le \|u_{jtt}\|^2 + C. \tag{3.35}$$

Using (3.31) and the above inequality, we obtain

$$\sum_{j=1}^{2}\|u_{jtt}\|^2 \le C\sum_{j=1}^{2}\int_0^t \|u_{jtt}\|^2\mathrm{d}s + C. \tag{3.36}$$

Using Gronwall inequality, we have

$$\sum_{j=1}^{2}\|u_{jtt}\|^2 \le C.$$

Then coming back to (3.35), the below inequality is true

$$\sum_{j=1}^{2}\|(-\triangle)^{\frac{\alpha}{2}}u_{jt}\|^2 \le C. \tag{3.37}$$

By (1.1), (1.2) and (3.36), we have

$$\begin{aligned}\|(-\triangle)^{\alpha}u_{jt}\| &\le C\|u_{jtt}\| + C\left\|\frac{\mathrm{d}}{\mathrm{d}t}(\beta(|u_1|^\rho + |u_2|^\rho)u_j)\right\| + C\left\|\frac{\mathrm{d}}{\mathrm{d}t}(k_ju_j)\right\| + \|f_{jt}\|\\ &\le C\sum_{j=1}^{2}\|u_j(x,0)\|_{H^{4\alpha}_{per}(\Omega)} + C\left\|\frac{\mathrm{d}}{\mathrm{d}t}(\rho(|u_1|^\rho + |u_2|^\rho)u_j)\right\| + C\left\|\frac{\mathrm{d}}{\mathrm{d}t}(k_ju_j)\right\| + \|f_{jt}\|.\end{aligned} \tag{3.38}$$

But

$$\begin{aligned}\left\|\frac{\mathrm{d}}{\mathrm{d}t}(\beta(|u_1|^\rho + |u_2|^\rho)u_j)\right\| &= \left\|\frac{\rho\beta}{2}|u_1|^{\rho-2}(u_1\bar{u}_{1t} + \bar{u}_1u_{1t})u_j + \frac{\rho\beta}{2}|u_2|^{\rho-2}(u_2\bar{u}_{2t} + \bar{u}_2u_{2t})u_j\right\|\\ &\le C(\|u_1(x,0)\|_{H^{2\alpha}_{per}(\Omega)} + \|u_2(x,0)\|_{H^{2\alpha}_{per}(\Omega)}).\end{aligned}$$

Then from (3.38) and the above inequality, we get (3.16). □

Lemma 3.5. *Suppose that α and ρ satisfy the conditions of Lemma 3.4, $u = (u_1, u_2)$ solves the problem (1.1)–(1.4). Then*

$$\sup_{0\le t<T}\sum_{j=1}^{2}\|(-\triangle)^{2\alpha}u_j\| \leqslant E_4, \tag{3.39}$$

where the constant E_4 depends on $\|u_j(x,0)\|_{H^{4\alpha}_{per}(\Omega)}$, $M_1, M_2, M_3, M_4, M_5, M_7, M_8$ and T.

Proof. When $\alpha > \max\{\frac{n}{2}, 1\}$, by (1.1), (1.2), (3.11) and (3.16), we obtain

$$\begin{aligned}\|(-\triangle)^{2\alpha}u_j\| &\le C\|(-\triangle)^{\alpha}u_{jt}\| + C\|(-\triangle)^{\alpha}((|u_1|^\rho + |u_2|^\rho)u_j)\| + C\|(-\triangle)^{\alpha}(k_ju_j)\| + C\|(-\triangle)^{\alpha}f_j\|\\ &\le C\|(-\triangle)^{\alpha}u_{jt}\| + C\|(-\triangle)^{[\alpha]+1}((|u_1|^\rho + |u_2|^\rho)u_j)\| + C\|(-\triangle)^{\alpha}(k_ju_j)\| + C\|(-\triangle)^{\alpha}f_j\|\\ &\le C + C\sum_{j=1}^{2}\|(-\triangle)^{[\alpha]+1}u_j\| + C\sum_{j=1}^{2}\|\nabla u_j\|^{2[\alpha]+2}_{L^{4[\alpha]+4}(\Omega)}.\end{aligned} \tag{3.40}$$

Let $\theta = \frac{[\alpha]+1}{2\alpha} < 1$, then

$$\frac{1}{2} - \frac{2[\alpha]+2}{n} = \theta\left(\frac{1}{2} - \frac{4\alpha}{n}\right) + \frac{1-\theta}{2}.$$

By Gagliardo–Nirenberg inequality and (3.1), we get

$$C\|(-\triangle)^{[\alpha]+1}u_j\| \le C\|(-\triangle)^{2\alpha}u_j\|^{\theta}\|u_j\|^{1-\theta} \le \frac{1}{4}\|(-\triangle)^{2\alpha}u_j\| + C. \tag{3.41}$$

Let $\delta = \frac{n+2}{4\alpha} < 1$, then

$$-\frac{1}{n} = \delta\left(\frac{1}{2} - \frac{2\alpha}{n}\right) + \frac{1-\delta}{2}.$$

By Gagliardo–Nirenberg inequality and (3.11), we obtain

$$\|\nabla u_j\|_{L^\infty(\Omega)} \le C\|u_j\|^{\delta}_{H^{2\alpha}_{per}(\Omega)} \le C. \tag{3.42}$$

Then

$$\|\nabla u_j\|_{L^{4[\alpha]+4}(\Omega)} \le C.$$

Using Gronwall inequality and (3.40), we have

$$\|(-\triangle)^{2\alpha}u_j\| \le C.$$

When $n = 1$ and $\frac{1}{2} < \alpha \le 1$, by (1.1), (1.2) and (3.16), we have

$$\begin{aligned}\|(-\triangle)^{2\alpha}u_j\| &\le C\|(-\triangle)^{\alpha}u_{jt}\| + C\|(-\triangle)^{\alpha}((|u_1|^{\rho} + |u_2|^{\rho})u_j)\| + C\|(-\triangle)^{\alpha}(k_ju_j)\| + C\|(-\triangle)^{\alpha}f_j\| \\ &\le C\sum_{j=1}^{2}\|u_j(x,0)\|_{H^{4\alpha}_{per}(\Omega)} + C\|\triangle u_j\| + C\sum_{j=1}^{2}\|\nabla|u_j|^2\| \\ &\le C\sum_{j=1}^{2}\|u_j(x,0)\|_{H^{4\alpha}_{per}(\Omega)} + C\|\triangle u_j\| + C\sum_{j=1}^{2}\|\nabla u_j\|^2_{L^4(\Omega)}.\end{aligned} \tag{3.43}$$

Let $\theta = \frac{2}{4\alpha} < 1$. Then

$$\frac{1}{2} = 2 + \theta\left(\frac{1}{2} - 4\alpha\right) + (1-\theta)\frac{1}{2}.$$

By Gagliardo–Nirenberg inequality and (3.1), we have

$$C\|\triangle u_j\| \le C\|(-\triangle)^{2\alpha}u_j\|^{\theta}\|u_j\|^{1-\theta} \le \frac{1}{4}\|(-\triangle)^{2\alpha}u_j\| + C. \tag{3.44}$$

Let $\delta = \frac{1}{16\alpha-4} < \frac{1}{4}$. Then

$$\frac{1}{4} = \delta\left(\frac{1}{2} - (4\alpha - 1)\right) + (1-\delta)\frac{1}{2}.$$

By Gagliardo–Nirenberg inequality and (3.11), we have

$$\begin{aligned}C\|\nabla u_j\|^2_{L^4(\Omega)} &\le C\|(-\triangle)^{2\alpha}u_j\|^{2\delta}\|\nabla u_j\|^{2(1-\delta)} \\ &\le C\|(-\triangle)^{2\alpha}u_j\|^{2\delta}\|(-\triangle)^{\alpha}u_j\|^{2(1-\delta)} \le \frac{1}{4}\|(-\triangle)^{2\alpha}u_j\| + C.\end{aligned} \tag{3.45}$$

Then, when $n = 1$ and $\frac{1}{2} < \alpha < 1$, by (3.43)–(3.45), we conclude that

$$\sum_{j=1}^{2}\|(-\triangle)^{2\alpha}u_j\| \le C(\|u_1(x,0)\|_{H^{4\alpha}_{per}(\Omega)} + \|u_2(x,0)\|_{H^{4\alpha}_{per}(\Omega)}).$$

Therefore, from (3.40) and the above inequality, we complete the proof of (3.39). □

4. Proof of the main result

Before proving Theorem 2.1, we first prove the existence of the weak solution to the problem (1.1)–(1.4) by using the Faedo–Galërkin method. We need the following lemmas.

Lemma 4.1. *Let B_0, B and B_1 be three Banach spaces. Assume that $B_0 \subset B \subset B_1$ and B_i, $i = 0, 1$ are reflexive. Suppose also that B_0 is compactly embedded in B. Let*

$$W=\left\{v|v\in L^{p_0}(0,T;B_0),\,v'=\frac{\mathrm{d}v}{\mathrm{d}t}\in L^{p_1}(0,T;B_1)\right\},$$

where T is finite and $1<p_i<\infty$, $i=0,1$. W is equipped with the norm

$$\|v\|_{L^{p_0}(0,T;B_0)}+\|v'\|_{L^{p_1}(0,T;B_1)}.$$

Then W is compactly embedded in $L^{p_0}(0,T;B)$.

Lemma 4.2. *Suppose that Q is a bounded domain in $\mathbb{R}^n_x\times\mathbb{R}_t$. $g_\mu,g\in L^q(Q)(1<q<\infty)$ and $\|g_\mu\|_{L^q(Q)}\le C$. Furthermore, suppose that:*

$$g_\mu\to g\quad \text{a.e. in } Q.$$

Then

$$g_\mu\rightharpoonup g\quad \text{weakly in } L^q(Q).$$

Lemma 4.3. *X is a Banach space. Suppose that $g\in L^p(0,T;X)$, $\frac{\partial g}{\partial t}\in L^p(0,T;X)(1\le p\le\infty)$. Then $g\in C([0,T],X)$ (after possibly being redefined on a set of measure zero).*

In the following, we prove the existence of weak solution to the problem (1.1)–(1.4).

Theorem 4.1. *Let $\alpha>\frac{n}{2}$, suppose that $0<\rho<\frac{4\alpha}{n}$. $u_1(x,0)\in H^\alpha_{per}(\Omega)$, $u_2(x,0)\in H^\alpha_{per}(\Omega)$. And assume that $k_j(x,t)\in C^2\cap H^{2\alpha}_{per}(\Omega)(j=1,2)$ and $k_{jt},k_{jtt}\in L^2(\Omega)$ $(j=1,2)$, $f_j(x,t)\in L^2(\Omega)\cap H^{2\alpha}_{per}(\Omega)$ and $f_{jt},f_{jtt}\in L^2(\Omega)$. Then there exists a function $u=(u_1,u_2)$ satisfying (1.1)–(1.4), such that*

$$u_j\in L^\infty(0,T;H^\alpha_{per}(\Omega)\cap L^{\rho+2}(\Omega)),\qquad u_{jt}\in L^\infty(0,T;H^{-\alpha}_{per}(\Omega)),\quad (j=1,2). \tag{4.1}$$

Proof. We prove Theorem 4.1 by three steps.

Step 1. Construction of approximate solutions by the Faedo–Galërkin method.

Fixing now a positive integer m, we will look for a function $u_{jm}=u_{jm}(t)$ of the form

$$u_{1m}(t)=\sum_{|k|=0}^{m}g_{km}(t)w_k,\qquad u_{2m}(t)=\sum_{|k|=0}^{m}h_{km}(t)w_k,\quad w_k=\mathrm{e}^{\mathrm{i}\langle k,x\rangle},\ k\in\mathbb{Z}^n, \tag{4.2}$$

where $g_{km}(t)$ and $h_{km}(t)(|k|=0,1,\dots,m)$ are selected by the following conditions

$$\begin{aligned}&(\mathrm{i}u_{jm,t},w_k)+((-\triangle)^\alpha u_{jm},w_k)+(\beta(|u_{1m}|^\rho+|u_{2m}|^\rho)u_{jm},w_k)+(k_j(x,t)u_{jm},w_k)=(f_j(x,t),w_k),\\&\quad 0\le|k|\le m,\ j=1,2\end{aligned} \tag{4.3}$$

and

$$u_{jm}(x,0)=u^j_{0m}\in[w_k,0\le|k|\le m],\qquad u^j_{0m}\to u^j_0(m\to\infty)\quad \text{in } H^\alpha_{per}(\Omega),\quad j=1,2. \tag{4.4}$$

Then (4.3) becomes the system of nonlinear ordinary differential equations subject to the initial conditions (4.4). According to standard existence theory for nonlinear ordinary differential equations, there exists a unique solution of (4.3) and (4.4) for a.e. $0\le t\le t_m$. By a priori estimates we obtain that $t_m=T$.

Step 2. A priori estimates.

As the proof of Lemmas 3.1 and 3.2, we have

$$u_{jm}\in L^\infty(0,T;H^\alpha_{per}(\Omega)\cap L^{\rho+2}(\Omega)),\quad j=1,2. \tag{4.5}$$

For $\forall\varphi_j$ $(j=1,2)\in H^\alpha_{per}(\Omega)$, we have

$$(\mathrm{i}u_{jm,t},\varphi_j)+((-\triangle)^\alpha u_{jm},\varphi_j)+(\beta(|u_{1m}|^\rho+|u_{2m}|^\rho)u_{jm},\varphi_j)+(k_j(x,t)u_{jm},\varphi_j)=(f_j(x,t),\varphi_j). \tag{4.6}$$

Then

$$\begin{aligned}|(u_{jm,t},\varphi_j)|&\le|((-\triangle)^\alpha u_{jm},\varphi_j)|+|(\beta(|u_{1m}|^\rho+|u_{2m}|^\rho)u_{jm},\varphi_j)|+|(k_j(x,t)u_{jm},\varphi_j)|+|(f_j(x,t),\varphi_j)|\\&\le C\|(-\triangle)^{\frac{\alpha}{2}}u_{jm}\|\|(-\triangle)^{\frac{\alpha}{2}}\varphi_j\|+C\|u_{1m}\|^\rho_{L^{\rho+2}(\Omega)}\|\varphi_j\|_{L^{\rho+2}(\Omega)}\|u_{jm}\|_{L^{\rho+2}(\Omega)}\\&\quad+C\|u_{2m}\|^\rho_{L^{\rho+2}(\Omega)}\|\varphi_j\|_{L^{\rho+2}(\Omega)}\|u_{jm}\|_{L^{\rho+2}(\Omega)}+C\|\varphi_j\|\\&\le C\|(-\triangle)^{\frac{\alpha}{2}}\varphi_j\|+C\|\varphi_j\|_{L^{\rho+2}(\Omega)}.\end{aligned} \tag{4.7}$$

By Sobolev embedding theorem, we have

$$\|\varphi_j\|_{L^{\rho+2}(\Omega)} \le C\|(-\triangle)^{\frac{\alpha}{2}}\varphi_j\|.$$

So by (4.6) and (4.7), we have

$$|(u_{jm,t}, \varphi_j)| \le C\|(-\triangle)^{\frac{\alpha}{2}}\varphi_j\|, \quad \forall \varphi_j \in H^{\alpha}_{per}(\Omega).$$

Therefore,

$$u_{jm,t} \in L^{\infty}(0, T; H^{-\alpha}_{per}(\Omega)), \quad j = 1, 2. \tag{4.8}$$

Step 3. Passaging to the limit.

By applying (4.5) and (4.8), we deduce that there exists a subsequence $u_{j\mu}$ from u_{jm}, such that

$$u_{j\mu} \rightharpoonup u_j \quad {}^*\text{-weakly in } L^{\infty}(0, T; H^{\alpha}_{per}(\Omega)), \tag{4.9}$$

$$u_{j\mu,t} \rightharpoonup u_{jt} \quad {}^*\text{-weakly in } L^{\infty}(0, T; H^{-\alpha}_{per}(\Omega)). \tag{4.10}$$

By (4.5), we have

$$u_{jm} \quad \text{is bounded in } L^2(0, T; H^{\alpha}_{per}(\Omega)). \tag{4.11}$$

By (4.8), we have

$$u_{jm,t} \quad \text{is bounded in } L^2(0, T; H^{-\alpha}_{per}(\Omega)). \tag{4.12}$$

Let

$$W = \{v | v \in L^2(0, T; H^{\alpha}_{per}(\Omega)),\ v_t \in L^2(0, T; H^{-\alpha}_{per}(\Omega))\}.$$

Since $H^{\alpha}_{per}(\Omega)$ is compactly embedded in $L^2(\Omega)$, W is compactly embedded in $L^2(0, T; L^2(\Omega))$ by Lemma 4.1. By (4.11) and (4.12), $u_{jm} \in W$. Then, there exists subsequence $u_{j\mu}$ which satisfies

$$u_{j\mu} \to u_j \quad \text{strongly in } L^2(0, T; L^2(\Omega)) \text{ and a.e.} \tag{4.13}$$

By using (4.5) and (4.13) and Lemma 4.2. we have

$$|u_{j\mu}|^{\rho}u_{j\mu} \rightharpoonup |u_j|^{\rho}u_j \quad {}^*\text{-weakly in } L^{\infty}(0, T; L^{\frac{\rho+2}{\rho+1}}(\Omega)). \tag{4.14}$$

Since $\forall \rho > 0$, $\exists\, k(k > 1, k \in N)$, such that $\rho > \frac{1}{k-1}$, then

$$|u_{i\mu}|^{\rho}u_{j\mu} \rightharpoonup |u_i|^{\rho}u_j \quad {}^*\text{-weakly in } L^{\infty}(0, T; L^{\frac{\rho+\frac{1}{k}}{\rho}}(\Omega)), (i \ne j). \tag{4.15}$$

Fixing j, k, by (4.3), we get

$$(\mathrm{i}u_{j\mu,t}, w_k) + ((-\triangle)^{\alpha}u_{j\mu}, w_k) + (\beta(|u_{1\mu}|^{\rho} + |u_{2\mu}|^{\rho})u_{j\mu}, w_k) + (k_j(x, t)u_{j\mu}, w_k) = (f_j(x, t), w_k),$$
$$j = 1, 2. \tag{4.16}$$

By applying (4.5), (4.14) and (4.15), we deduce that there exists a subsequence $u_{j\mu}$ from u_{jm}, such that

$$((-\triangle)^{\alpha}u_{j\mu}, w_j) \rightharpoonup ((-\triangle)^{\alpha}u_j, w_j) \quad {}^*\text{-weakly in } L^{\infty}(0, T),$$
$$(u_{j\mu,t}, w_j) \rightharpoonup (u_{jt}, w_j) \quad {}^*\text{-weakly in } L^{\infty}(0, T),$$
$$(\beta|u_{j\mu}|^{\rho}u_{j\mu}, w_j) \rightharpoonup (\beta|u_j|^{\rho}u_j, w_j) \quad {}^*\text{-weakly in } L^{\infty}(0, T),$$
$$(\beta|u_{i\mu}|^{\rho}u_{j\mu}, w_j) \rightharpoonup (\beta|u_i|^{\rho}u_j, w_j)\ (i \ne j) \quad {}^*\text{-weakly in } L^{\infty}(0, T).$$

Then from (4.16), we have

$$(\mathrm{i}u_{jt}, w_k) + ((-\triangle)^{\alpha}u_j, w_k) + (\beta(|u_1|^{\rho} + |u_2|^{\rho})u_j, w_k) + (k_ju_j, w_k) = (f_j, w_k), \quad j = 1, 2.$$

The above equality holds for any fixed k. By the density of the basis $w_k(k \in \mathbb{Z}^n)$, we have

$$(\mathrm{i}u_{jt}, v) + ((-\triangle)^{\alpha}u_j, v) + (\beta(|u_1|^{\rho} + |u_2|^{\rho})u_j, v) + (k_ju_j, v) = (f_j, v), \quad \forall v \in H^{\alpha}_{per}(\Omega).$$

Hence u satisfies (1.1), (1.2), (4.1) and (4.2). By (4.5) and (4.8) and Lemma 4.3, we obtain that $u_{j\mu} \in C([0, T], H^{-\alpha}_{per}(\Omega))$. Then

$$u_{j\mu}(x, 0) \rightharpoonup u_j(x, 0) \quad \text{weakly in } H^{-\alpha}_{per}(\Omega).$$

But from (4.4), we have

$$u_{j\mu}(x,0) \to u_0^j \quad \text{in } H^{\alpha}_{per}(\Omega).$$

Therefore $u_j(x,0) = u_0^j$. $\square$

Theorem 4.1 generalizes the result of the global existence of weak solution to the nonlinear Schrödinger equations in [10]. Now we prove our main results.

By a priori estimates in Lemmas 3.1–3.5 and Theorem 4.1, we derive a global smooth solution $u = (u_1, u_2)$ of the problem (1.1)–(1.4), such that

$$u_j \in L^{\infty}(0,T; H^{4\alpha}_{per}(\Omega)), \qquad u_{jt} \in L^{\infty}(0,T; H^{2\alpha}_{per}(\Omega)).$$

We prove the uniqueness of the solution to the problem (1.1)–(1.4) in the following. Suppose that $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are two solutions which satisfy the problem (1.1)–(1.4). Then $r = (r_1, r_2) = u - v$ satisfies

$$ir_{jt} + (-\triangle)^{\alpha} r_j + \beta((|u_1|^{\rho} + |u_2|^{\rho})u_j - (|v_1|^{\rho} + |v_2|^{\rho})v_j) + k_j r_j = 0, \tag{4.17}$$

with $r_j(x,0) = 0$.

Take the inner product of (4.17) with r_j to get

$$i(r_{jt}, r_j) + ((-\triangle)^{\alpha} r_j, r_j) + \beta((|u_1|^{\rho} + |u_2|^{\rho})u_j - (|v_1|^{\rho} + |v_2|^{\rho})v_j, r_j) + (k_j r_j, r_j) = 0. \tag{4.18}$$

Consider the imaginary part of (4.18) to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|r_j\|^2 + \mathrm{Im}\beta((|u_1|^{\rho} + |u_2|^{\rho})u_j - (|v_1|^{\rho} + |v_2|^{\rho})v_j, r_j) = 0. \tag{4.19}$$

But

$$\begin{aligned}
\mathrm{Im}\beta((|u_1|^{\rho} + |u_2|^{\rho})u_j - (|v_1|^{\rho} + |v_2|^{\rho})v_j, r_j)| &\le C|((|u_1|^{\rho} + |u_2|^{\rho})(u_j - v_j) + (|u_1|^{\rho} + |u_2|^{\rho} - |v_1|^{\rho} - |v_2|^{\rho})v_j, r_j) \\
&\le C\sum_{j=1}^{2} |u_j|^{\rho}\|u_j - v_j\|^2 + C|((|u_1|^{\rho} + |u_2|^{\rho} - |v_1|^{\rho} - |v_2|^{\rho})v_j, r_j)| \\
&\le C\|r_j\|^2.
\end{aligned}$$

By the above inequality and Gronwall's inequality, we have $\|r_j\| = 0$. So $r = 0$.

Therefore, we complete the proof of Theorem 2.1.

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